

# SYSTEMS OF DILATED FUNCTIONS: COMPLETENESS, MINIMALITY, BASISNESS

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ABSTRACT. We discuss completeness, minimality, and basisness, in  $L^2[0, \pi]$  and  $L^p[0, \pi]$ ,  $p \neq 2$ , of dilated systems  $u_n(x) = S(nx)$ ,  $n \in \mathbb{N}$ , where  $S$  is a trigonometric polynomial

$$S(x) = \sum_{k=0}^m a_k \sin(kx), \quad a_0 a_m \neq 0.$$

We will present some results and mention a few unsolved questions.

We announce a series of results which complement and extend the research of [1] – [4]. The proofs will be given elsewhere.

Define the polynomial

$$a(z) = \sum_{j=0}^m a_j z^j,$$

and sets

$$\begin{aligned} Z(a) &= F^- \cup F^0 \cup F^+ \\ a(\alpha) = 0 &\quad |\alpha| < 1 \quad |\alpha| = 1 \quad |\alpha| > 1. \end{aligned}$$

The isometry  $T : f(x) \rightarrow f(2x)$  has spectrum  $\sigma(T) = \overline{\mathcal{D}}$ ,  $\mathcal{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ . Then  $u_n = a(T)\{\sin(nx)\}$ . We factorize  $a(z) = a^-(z)a^0(z)a^+(z)$ . We note that  $a^+(T)$  is invertible, for letting  $B = (a^+(T))^{-1}$ ,

$$B = \prod_{\alpha \in F^+} (\alpha - T)^{-1} = \frac{1}{2\pi i} \int_{|z|=1+\delta} R(z, t) \frac{dz}{a^+(z)},$$

$$1 + 2\delta = \min\{|\alpha| : \alpha \in F^+\}.$$

Let  $v_n = Bu_n = a^-(T)a^0(T)\{\sin(nx)\}$ .

**Claim 1.**

- (a) The system  $U = \{u_n\}$  is a basis in  $L^2[0, \pi]$  if and only if  $F^- \cup F^0 = \emptyset$ .
- (b) If  $F^- \neq \emptyset$  the system is not complete.
- (c) For any  $a$  the system  $U$  is minimal.

So assume  $(\star) F = F^0 \neq \emptyset$ , i.e., all roots  $\alpha$ ,  $a(\alpha) = 0$ , are in  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .

**Claim 2.** Under  $(\star)$

$$a(z) = a_m \prod_{\alpha \in F^0} (\alpha - z)^{\mu(\alpha)}.$$

Put  $\kappa^* = \max\{\mu(\alpha) - 1 : \alpha \in F^0\}$ . The system  $U$  is complete, and minimal, i.e.,  $\exists\{\Phi_k\}$ ,  $\langle \Phi_k, u_n \rangle = \delta_{kn}$ , and

$$\|\Phi_k\|_q \asymp (\log k)^{\kappa^* + 1/2},$$

so  $U$  is not a basis in  $L^2$  (or  $L^p$ ,  $1 < p < \infty$ ).

Now we go to the “multi-frequency” case. We again set  $U = \{u_n(x)\}$  and  $u_n(x) = S(nx)$ ,  $0 \leq x \leq 2\pi$ ,

$$\begin{aligned} S(x) &= \sum_{j \in J} a_j \exp(ijx), \quad |J| < \infty \\ &= \sum_{\substack{\alpha \in \mathbb{N}_0^m \\ \alpha \in K, \quad K \subseteq \mathbb{N}_0^m}} a(\alpha) \exp \left( i \left[ \prod_{j=1}^m p_j^{\alpha_j} \right] x \right), \quad |K| < \infty, \end{aligned}$$

where  $\{p_j\}_{j=1}^m$  is a set of primes. Put

$$A(\omega) = \sum_{\alpha \in K} a(\alpha) w^\alpha, \quad w^\alpha = \prod_{j=1}^m w_j^{\alpha_j}.$$

and

$$\begin{aligned} \mathbb{N}(\omega) &= \{\omega \cdot p^\alpha : \alpha \in \mathbb{N}_0^m\} \\ \omega \in \Omega &= \{q \in \mathbb{N} : q \text{ does not have factors } p_j, \quad 1 \leq j \leq m\}. \end{aligned}$$

Let

$$T_j : f(x) \mapsto f(p_j x), \quad 1 \leq j \leq m.$$

Consider

$$E \cong \ell^2 \simeq H^2(\mathcal{D}) \simeq \ell^2(\Omega; H^2(\mathcal{D}^m)).$$

All  $E(\omega) = \text{Im } Q(\omega)$ ,  $\omega \in \Omega$ , are invariant with respect to the isometries  $T_j$ ,  $1 \leq j \leq m$ , i.e. multiplication by  $w_j$  in  $H^2(\mathcal{D}^m)$ . Certainly,

$$\|f\|^2 = \sum_{\omega \in \Omega} \|Q(\omega)f\|^2.$$

Now all the questions about  $U$  become the questions about the system

$$V = \{v(\alpha)\}_{\alpha \in \mathbb{N}_0^m}; \quad v(\alpha)(w) = A(w)w^\alpha \quad \text{in } H^2(\mathcal{D}^m).$$

**Claim 3.** *If*

$$(\star\star) \quad Z(A) \cap \overline{\mathcal{D}^m} = \emptyset,$$

*then  $A(T)^{-1} = B$  is well-defined and  $\{v(\alpha)\}$  is a (Riesz) basis and  $U$  is a (Riesz) basis as well.*

*If  $V$  (or  $U$ ) is a Riesz basis then  $(\star\star)$  holds.*

**Claim 4.** *The system  $U$  is minimal if  $a(0) \neq 0$ .*

Indeed, with

$$\langle f, g \rangle = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} f(w)g(w) d^m t, \quad w_j = e^{it_j}, \quad 1 \leq j \leq m,$$

(no bar, no conjugation),  $\langle w^{-\tau}, w^\alpha \rangle = \delta(\alpha, \tau)$ ,  $\forall \alpha, \tau \in \mathbb{Z}^m$ . Then

$$\frac{1}{A(w)} = \sum_{\sigma \in \mathbb{N}_0^m} b(\sigma) w^\sigma, \quad \frac{w^{-\tau}}{A(w)} = \sum_{\sigma \in \mathbb{N}_0^m} b(\sigma) w^{\sigma - \tau},$$

but if  $\sigma - \tau \leq 0$  does not hold then

$$\langle w^{\sigma - \tau}, w^\alpha \rangle = 0, \quad \forall \alpha \in \mathbb{N}_0^m,$$

so put  $\Phi_t(w) = \sum_{\sigma \leq \tau} b(\sigma)w^{\sigma-\tau}$ . This *finite* sum is well-defined, and

$$\langle \Phi_\tau, v(\alpha) \rangle = \delta(\alpha, \tau), \quad \text{for all } \alpha, \tau \in \mathbb{N}_0^m.$$

$$Z(A) \cap \overline{\mathcal{D}^m} = Z(A) \cap \mathbb{T}^m.$$

Completeness implies uniqueness of the system  $\Phi_\tau$  and the fact that for 1D projections  $P_\tau = \langle \bullet, \Phi_\tau \rangle v(\tau)$ ,

$$\|P_\tau\| = \|\Phi_\tau\| \cdot \|v(\tau)\| \asymp \|\Phi_\tau\|.$$

But

$$\|\Phi_\tau\|^2 = \sum_{\sigma \leq \tau} |b(\sigma)|^2, \quad B(w) = \frac{1}{A(w)},$$

so these norms are uniformly bounded if and only if

$$\frac{1}{A(w)} \in H^2(\mathcal{D}^m), \text{ or } \frac{1}{P(t)} \in L^1(\mathbb{T}^m), \text{ where } P(t) = |A(e^{it})|^2.$$

**Claim 5.** *If  $m \leq 3$  and  $Z(A) \cap \overline{\mathcal{D}^m} \neq \emptyset$ , then*

$$\frac{1}{A(w)} \notin H^2(\mathcal{D}^m).$$

*Under the same assumptions,  $V$  or  $U$  is NOT a basis.*

For  $m \geq 4$ , it could happen that  $\frac{1}{A(w)} \in H^2(\mathcal{D}^m)$ .

**Example 6.** Fix  $c_k > 0$ ,  $1 \leq k \leq m$ , with  $\sum_{k=1}^m c_k = 1$ , and define

$$(E^*) \quad A(w) = 1 - \sum_{k=1}^m c_k w_k.$$

Then

$$\begin{aligned} P(t) &= \left| \sum_{k=1}^m c_k 2 \sin^2 \left( \frac{t_k}{2} \right) \right|^2 + \left| \sum_{k=1}^m c_k \sin(t_k) \right|^2 \\ &\asymp r^4 + |\ell(t)|^2, \quad \ell(t) = \sum_{k=1}^m c_k t_k, \quad |t| \ll 1. \end{aligned}$$

We note that  $\int_{|\zeta| \leq \delta} \frac{d\zeta_0 d\zeta_1 \dots d\zeta_{m-1}}{\zeta_0^2 + \zeta_1^4 + \dots + \zeta_{m-1}^4} < \infty$  if and only if  $m \geq 4$ , since it is within a constant multiple of

$$\int_0^\delta \int_0^{\rho^2} \frac{d\zeta \rho^{m-2} d\rho}{\zeta^2 + \rho^4} = \int_0^\delta \int_0^1 \frac{d\eta \rho^{m-4} d\rho}{1 + \eta^2}.$$

Recall that  $v(\alpha) = A(w)w^\alpha$ ,  $\alpha \in \mathbb{N}_0^m$ . Instead of asking whether this system is a basis in  $H^2(\mathcal{D}^m)$ , or  $\ell^2(\mathbb{N}_0^m)$  we can move to weighted  $H^2(\mathcal{D}^m; P)$ , or  $L^2(\mathbb{T}^m; P)$  and ask whether  $\{w^\alpha\}$  is a basis.

**Claim 7.** *In the case  $(E^*)$ ,  $m \geq 4$ , for the partial sums  $\Sigma(\tau)f = \sum_{\alpha \leq \tau} \langle \Phi_\alpha, f \rangle v_\alpha$ , the norms  $\|\Sigma(\tau)\|$  are not bounded.*

The proof is based on the multi-dimensional  $A_2$  Muckenhoupt condition ([5], [6]).  
But if we go to the original system

$$v_\alpha(x) = \sum_{k=0}^K a_k \exp(i[p^\alpha]kx), \quad p = (p_j)_{j=1}^m, \quad \alpha \in \mathbb{N}_0^m,$$

its linear ordering would fit to a monotone arrangement of the multi-index sequence  $\{p^\alpha\}$ , or its linear ordering by monotonicity of the linear form  $M(\alpha) = \sum_{j=1}^m \alpha_j \log p_j$ . It leads us to the question on the boundedness of the projection  $Q_M$ .

$$\begin{aligned} \exp i(\alpha, t) &\rightarrow \text{the same, if } M(\alpha) \geq 0 \\ &\rightarrow 0 \text{ if } M(\alpha) < 0. \end{aligned}$$

in the weighted  $L^2(\mathbb{T}^m; P)$ ,  $m \geq 4$ , say,

$$P(t) = \left( \sum_{j=1}^m t_j \right)^2 + \left( \sum_{j=1}^m t_j^2 \right)^2.$$

$M(\alpha)$  is “essentially irrational,” i.e., the coefficients  $\mu_j^* = \log p_j$ ,  $1 \leq j \leq m$ , of the linear function  $M(y) = \sum_{j=1}^m \mu_j y_j$  are rationally independent.

If all  $\mu_j$  were rational  $Q_M$  would be equivalent to the case  $M_0(y) = y_1$ ; then known  $A_2$  conditions are applicable and  $Q_{M_0}$  and  $Q_M$  are bounded, for any  $m$ .

If  $Q_{M^*}$  were bounded,  $m \geq 4$  (I do not believe so) we would have Babenko–type Schauder (but not Riesz) basis in  $H^2(\mathcal{D}^m)$ , or even in  $L^2(\mathbb{T}^m)$ .

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